ESC103 - ENGINEERING MATHEMATICS and COMPUTATION

Independence, Dependence, and Column Space

In example 1 of unit 7, the column vectors of matrix A are called independent vectors because each vector is a vector that cannot be expressed as a combination of the other column vectors.

A more precise definition is the column vectors are independent when the only combination that produces

$$A_1\vec{x} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

is when

$$\vec{x} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Example 1:

$$A_1\vec{x} = \begin{bmatrix}1 & 0 & 0\\2 & 4 & 0\\3 & 5 & 6\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

$$= x_1\begin{bmatrix}1\\2\\3\end{bmatrix} + x_2\begin{bmatrix}0\\4\\5\end{bmatrix} + x_3\begin{bmatrix}0\\0\\6\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Solving for the x's:

$$x_1(1) + x_2(0) + x_3(0) = 0 \quad \therefore x_1 = 0$$
$$x_1(2) + x_2(4) + x_3(0) = 0 \quad \therefore x_2 = 0$$
$$x_1(3) + x_2(5) + x_3(6) = 0 \quad \therefore x_3 = 0$$

$\therefore$ The column vectors of A_1 are independent.

Example 2:

$$A_2 = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 5 \\ 6 & 0 & 6 \end{bmatrix} \quad \vec{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

$$A_2 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ The column vectors of A_2 are not independent
$\therefore$ These column vectors are <u>dependent</u>

Note the zero-vector is dependent.

Example 3:

$$A_3 = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \\ 5 & 15 & 20 \end{bmatrix} \quad \vec{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

$$A_3 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ The column vectors of A_3 are dependent

Column Space

The column space of matrix A is the set of all vectors $A\vec{X}$, ie. All linear combinations of the column vectors of matrix A.

Example 1: Column space of A_1 is all R^3
Example 2: Column space of A_2 is a 2-D plane in R^3
Example 3: Column space of A_3 is a 1-D line in R^3

For all of R^3, you need 3 independent column vectors.
For a 2-D plane through the origin, you need 2 independent column vectors.
For a 1-D line through the origin, you need 1 independent column vector.

Example 4:

$$A_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

All linear combinations of the column vectors produce

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which is just a point at the origin in R^3.

Rank of a Matrix

The number of columns of matrix A that are independent is the rank of the matrix.

Example:

rank $A_1 = 3$
rank $A_2 = 2$
rank $A_3 = 1$
rank $A_4 = 0$