ESC103 - ENGINEERING MATHEMATICS and COMPUTATION

More About Matrices

Given matrix $(m \times n)$

If $m=n$, then the matrix is "square"

Matrix addition ($A+B \rightarrow$ matrices must be dimensionally consistent) and scalar multiplication work the <u>same</u> way as with vectors.

We often think of the columns of matrix A as vectors:

$$\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n$$

Each of these vectors is in m-dimensional space.

Example:

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \uparrow & \uparrow & \uparrow & \uparrow \\ \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{a}_4 \\ \downarrow & \downarrow & \downarrow & \downarrow \end{bmatrix}$$

$$\vec{a}_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \quad \vec{a}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \vec{a}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad \vec{a}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Typically when we multiply a matrix by a vector, we say:

$A\vec{x}$ where A is the matrix
$\vec{x}$ is the vector

Example:

$$A\vec{x} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Row Picture:

$$\begin{bmatrix} -X_1 + X_2 \\ -X_2 + X_3 \\ -X_3 + X_4 \end{bmatrix}$$

Column Picture:

$$X_1\begin{bmatrix}-1\\0\\0\end{bmatrix} + X_2\begin{bmatrix}1\\-1\\0\end{bmatrix} + X_3\begin{bmatrix}0\\1\\-1\end{bmatrix} + X_4\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$= \begin{bmatrix} -X_1 + X_2 \\ -X_2 + X_3 \\ -X_3 + X_4 \end{bmatrix}$$

System of Equations (Solved 3 Ways)

Example:

$$\begin{aligned} X_1 &= b_1 \\ 2X_1 + 4X_2 &= b_2 \\ 3X_1 + 5X_2 + 6X_3 &= b_3 \end{aligned}$$

Unknowns: X_1, X_2, X_3

Constants: b_1, b_2, b_3

$$\begin{bmatrix}1&0&0\\2&4&0\\3&5&6\end{bmatrix}\begin{bmatrix}X_1\\X_2\\X_3\end{bmatrix} = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix} \qquad A_1 = \begin{bmatrix}1&0&0\\2&4&0\\3&5&6\end{bmatrix} \quad \vec{b} = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix} \qquad A_1\vec{X} = \vec{b}$$

lower triangle matrix

Method 1: "Algebraically"

1st Equation: $X_1 = b_1$ ✓

2nd Equation: $X_2 = \frac{1}{4}(b_2 - 2X_1) = \frac{1}{4}(b_2 - 2b_1)$ ✓

3rd Equation: $X_3 = \frac{1}{6}(b_3 - 3X_1 - 5X_2) = \frac{1}{6}(b_3 - \frac{1}{2}b_1 - \frac{5}{4}b_2)$ ✓

Method 2: Row Picture

$$\left.\begin{aligned} X_1 + 0X_2 + 0X_3 &= b_1 \\ 2X_1 + 4X_2 + 0X_3 &= b_2 \\ 3X_1 + 5X_2 + 6X_3 &= b_3 \end{aligned}\right\} \text{Solution is the intersection of these 3 planes}$$

Method 3: Column Picture

$$X_1\begin{bmatrix}1\\2\\3\end{bmatrix} + X_2\begin{bmatrix}0\\4\\5\end{bmatrix} + X_3\begin{bmatrix}0\\0\\6\end{bmatrix} = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

$X_1\begin{bmatrix}1\\2\\3\end{bmatrix} + X_2\begin{bmatrix}0\\4\\5\end{bmatrix}$ are not parallel

consider:

$$X_1\begin{bmatrix}1\\2\\3\end{bmatrix} + X_2\begin{bmatrix}0\\4\\5\end{bmatrix} + X_3\begin{bmatrix}0\\0\\6\end{bmatrix}$$

is $\begin{bmatrix}0\\0\\6\end{bmatrix}$ in the plane formed by combinations of $\begin{bmatrix}1\\2\\3\end{bmatrix}$ and $\begin{bmatrix}0\\4\\5\end{bmatrix}$

Can you find $c\begin{bmatrix}1\\2\\3\end{bmatrix} + d\begin{bmatrix}0\\4\\5\end{bmatrix} = \begin{bmatrix}0\\0\\6\end{bmatrix}$?

$C(1) + d(0) = C = 0$

$C(2) + d(4) = 4d = 0 \Rightarrow d = 0$

$C(3) + d(5) = 0 \neq 6$

$\therefore \begin{bmatrix}0\\0\\6\end{bmatrix}$ does not lie in the plane formed by combinations of $\begin{bmatrix}1\\2\\3\end{bmatrix}$ and $\begin{bmatrix}0\\4\\5\end{bmatrix}$

$\therefore$ All combinations of the column vectors of matrix A fill the $\mathbb{R}^3$ space

$\therefore$ For any values of $\vec{b}$, we can find values for $\vec{x}$ that solve the system