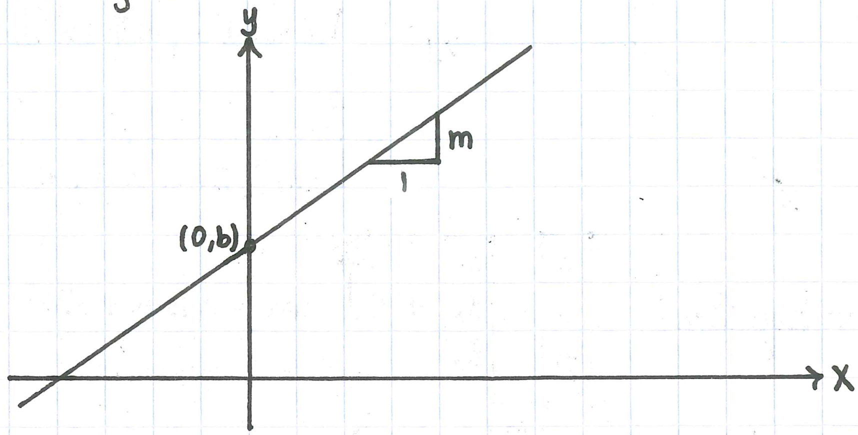
ESC103 - ENGINEERING MATHEMATICS and COMPUTATION

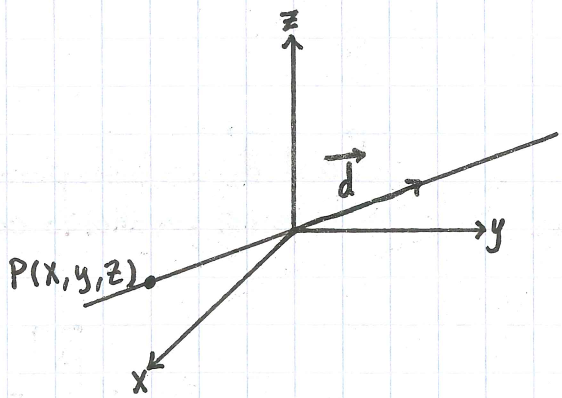
Lines and Planes in 3-D (R^3)

Consider $y = mx + b$



$y = mx + b$ is really a collection of points. Another way of thinking about $y = mx + b$ is as a mapping. In other words, from one value of x, you can calculate a coresponding value of y.

Lines in 3-D Through the Origin

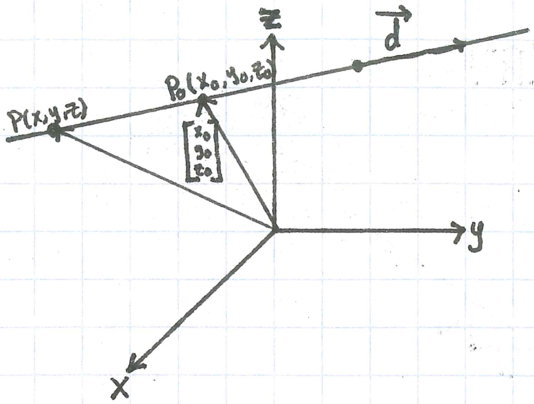


$\vec{d}$ is a direction vector parallel to the line.

P is any point on the line.

Vector Equation: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = c\vec{d}$ where c is a scalar

Lines in 3-D Not Through the Origin



Scalar multiples of $\vec{d}$ do not represent this line.
We need a point on the line.

$P_0(X_0, Y_0, Z_0)$ is a known point on the line.
$P(X, Y, Z)$ is any point on the line.

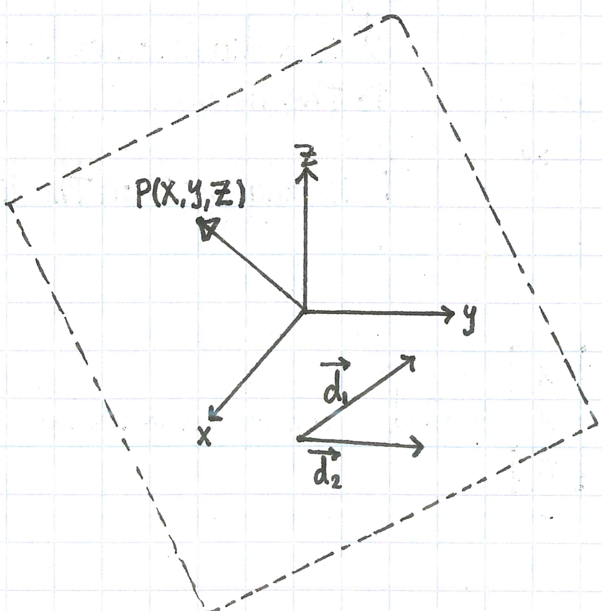
Vector Equation: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + c\vec{d}$ where c is a scalar

Linear combinations of $\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$ and $c\vec{d}$ represent this line.

Recall $y = mx + b$. Can we come up with a vector representation of $y = mx + b$?

$P_0 = (0, b)$ $\quad \vec{d} = \begin{bmatrix} 1 \\ m \end{bmatrix}$ $\quad$ Vector Representation: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ b \end{bmatrix} + c\begin{bmatrix} 1 \\ m \end{bmatrix}$

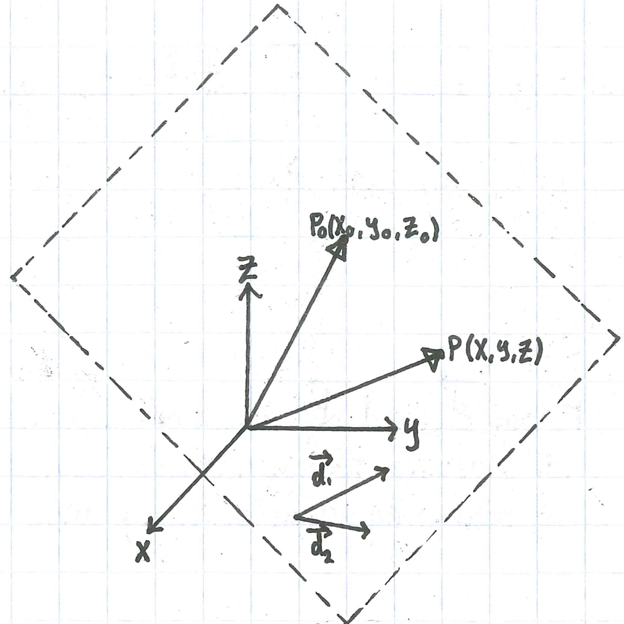
Planes Through the Origin



$\vec{d_1}$ and $\vec{d_2}$ are both parallel to the plane and are not parallel to each other.

Vector Equation: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = c\vec{d_1} + d\vec{d_2}$ where c and d are scalars

Planes in 3-D Not Through the Origin



$\vec{d_1}$ and $\vec{d_2}$ are both parallel to the plane and are not parallel to eachother.

$P_0(x_0, y_0, z_0)$ is a known point on the plane.

$P(x, y, z)$ is any point on the plane.

Vector Equation: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + c\vec{d_1} + d\vec{d_2}$ where c and d are scalars

We will take advantage of the fact that we can find a vector, called a normal vector $\vec{n}$, that is orthogonal to all vectors parallel to the plane.

Let $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ Where a, b, and c are scalars

$\therefore \overrightarrow{P_0P} \cdot \vec{n} = 0$

$P_0(x_0, y_0, z_0) \qquad P(x, y, z)$

$$\begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Any and all (x, y, z) that satisfy this equation lie in the plane.

$$ax + by + cz = ax_0 + by_0 + cz_0$$

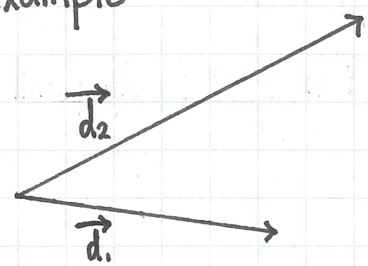
let $d = ax_0 + by_0 + cz_0$

$ax + by + cz = d$ where a, b, c, and d are scalars

Example: What do you know about $3x + 2y + 1z = 0$

1. It is a plane
2. It goes through the origin because $\begin{bmatrix}0\\0\\0\end{bmatrix}$ satisfies the equation.
3. $\vec{n} = \begin{bmatrix}3\\2\\1\end{bmatrix}$ Since $a=3$, $b=2$, $c=1$ and $\vec{n} = \begin{bmatrix}a\\b\\c\end{bmatrix}$

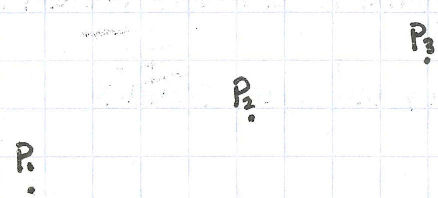
Example:



From $\vec{d_1}$ and $\vec{d_2}$ which are parallel to the plane and not parallel to each other, can you come up with the scalar equation of the plane?

$\vec{d_1}$ and $\vec{d_2}$ will give you a normal vector if you take the cross product between them. However, you still need a point on the plane.

Example:



Can you get the equation of a plane through these points?

No! You can get two direction vectors, but they will be paralell to eachother.