ESC103-ENGINEERING MATHEMATICS and COMPUTATION

Multiplying Vectors Using the Dot Product

Let $\vec{V} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$ and $\vec{W} = \begin{bmatrix} W_1 \\ W_2 \\ W_3 \end{bmatrix}$

Definition of the Dot Product (Scalar Product)

$$\vec{V}\cdot\vec{W} = V_1W_1 + V_2W_2 + V_3W_3 \quad \text{(produces a scalar)}$$

Properties of the Dot Product

P1: $\vec{V}\cdot(\vec{W}+\vec{Z}) = \vec{V}\cdot\vec{W} + \vec{V}\cdot\vec{Z}$
- Distributive Law

P2: $\vec{V}\cdot\vec{W} = \vec{W}\cdot\vec{V}$
- Commutative Law

P3: $C(\vec{V}\cdot\vec{W}) = (C\vec{V})\cdot\vec{W} = \vec{V}\cdot(C\vec{W})$ Where C is a scalar

Connection Between Length and Dot Product

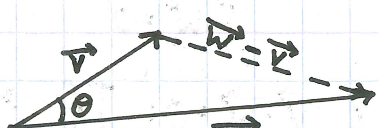
$$\|\vec{V}\| = \sqrt{V_1^2+V_2^2+V_3^2}$$

$$\therefore \|\vec{V}\|^2 = V_1^2+V_2^2+V_3^2$$

$$\|\vec{V}\|^2 = \vec{V}\cdot\vec{V}$$

$$\therefore \boxed{\|\vec{V}\| = \sqrt{\vec{V}\cdot\vec{V}}}$$

Application of the Dot Product (Finding the Angle Between Two Vectors)



Recall $\|\vec{V}\|^2 = \vec{V}\cdot\vec{V}$

$$\|\vec{W}-\vec{V}\|^2 = (\vec{W}-\vec{V})\cdot(\vec{W}-\vec{V})$$

$$= \vec{W}\cdot\vec{W} - \vec{W}\cdot\vec{V} - \vec{V}\cdot\vec{W} + \vec{V}\cdot\vec{V}$$

$$\|\vec{W}-\vec{V}\|^2 = \|\vec{W}\|^2 + \|\vec{V}\|^2 - 2\vec{W}\cdot\vec{V} \quad ①$$

Recall Cosine Law ($c^2 = a^2 + b^2 - 2ab\cos\theta$)

$$\|\vec{W}-\vec{V}\|^2 = \|\vec{W}\|^2 + \|\vec{V}\|^2 - 2\|\vec{W}\|\|\vec{V}\|\cos\theta \quad ②$$

Note: $\frac{1}{\|\vec{W}\|}\vec{W}$ and $\frac{1}{\|\vec{V}\|}\vec{V}$ are both unit vectors.

Compare ① and ②

$$\therefore \vec{W}\cdot\vec{V} = \|\vec{W}\|\|\vec{V}\|\cos\theta \Rightarrow \boxed{\cos\theta = \frac{\vec{W}\cdot\vec{V}}{\|\vec{W}\|\|\vec{V}\|} \quad \vec{W}\neq\vec{0},\ \vec{V}\neq\vec{0}}$$

Recall $\vec{w}\cdot\vec{v} = \|\vec{w}\|\|\vec{v}\|\cos\theta$. Note that $\|\vec{w}\|$ and $\|\vec{v}\|$ will be positive and $\cos\theta$ will be between -1 and $+1$. This tells us something surprising about the angle between $\vec{w}$ and $\vec{v}$:

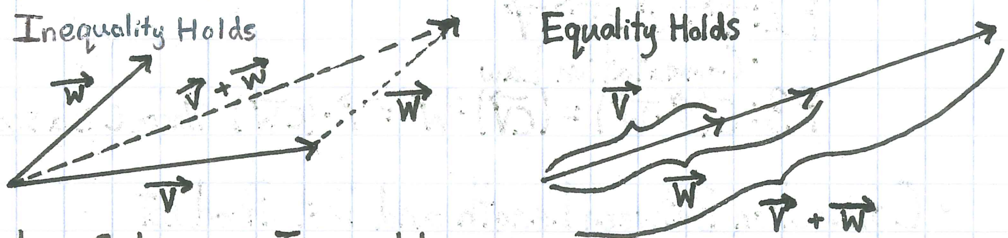
$$\vec{w}\cdot\vec{v} > 0 \qquad 0 \le \theta < \tfrac{\pi}{2} \text{ (acute)}$$
$$\vec{w}\cdot\vec{v} < 0 \qquad \tfrac{\pi}{2} < \theta \le \pi \text{ (obtuse)}$$
$$\vec{w}\cdot\vec{v} = 0 \qquad \theta = \tfrac{\pi}{2} \text{ (right)}$$

Two Important Inequalities

Triangle Inequality

$$\|\vec{v}+\vec{w}\| \le \|\vec{v}\| + \|\vec{w}\| \quad \text{Proof Assigned}$$

Consider:



Cauchy-Schwarz Inequality

$$|\vec{v}\cdot\vec{w}| \le \|\vec{v}\|\|\vec{w}\| \quad \text{Proof Assigned}$$

Hint: Use the fact that $0 \le |\cos\theta| \le 1$

For two nonzero vectors $\vec{v}$ and $\vec{w}$

$$\vec{v}\cdot\vec{w} = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \tfrac{\pi}{2}$$

If $\vec{v}$ and/or $\vec{w}$ is the zero vector:

$$\vec{v}\cdot\vec{w} = 0$$

Formal Definition of $\vec{v}\cdot\vec{w} = 0$: If $\vec{v}\cdot\vec{w} = 0$, then we say $\vec{v}$ and $\vec{w}$ are orthogonal.

Note: The zero vector is orthogonal to every non zero vector including itself.