ESC103 - ENGINEERING MATHEMATICS and COMPUTATION

Inverse Matrices

Here we will only be working with square matrices $(n \times n)$.

Definition: The matrix A is invertible if there exists another matrix, denoted as A^{-1}, such that:

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I$$

Note 1: If Matrix A is invertible, it cannot have more than one inverse.

Suppose Matrices B and C satisfy $BA = I = (AB)$ and $AC = I = (CA)$ then:

$$B(AC) = (BA)C$$
$$B(I) = (I)C$$

$\therefore B = C$
$\therefore B$ and C must be the same

Note 2: If Matrix A is invertible, there is one and only one solution to $A\vec{x} = \vec{b}$, and that solution is given by:

$$A\vec{x} = \vec{b}$$
$$(A^{-1}A)\vec{x} = A^{-1}\vec{b}$$
$$I\vec{x} = A^{-1}\vec{b}$$
$$\vec{x} = A^{-1}\vec{b}$$

Note 3: If Matrix A is invertible, then $A\vec{x} = \vec{0}$ can only have the trivial solution:

$$A\vec{x} = \vec{0}$$
$$(A^{-1}A)\vec{x} = A^{-1}\vec{0}$$
$$I\vec{x} = \vec{0}$$

$\therefore \vec{x} = \vec{0}$
$\therefore$ If there is a nonzero vector $\vec{x}$ such that $A\vec{x} = \vec{0}$, then Matrix A is not invertible.

Note 4: Matrix A is invertible if and only if its RREF is the identity matrix.

Recall $A = CR$

Let's say Matrix is 3x3

$$A = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \text{Col 1} & \text{Col 2} & \text{Col 3} \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$$

If col 1, col 2, col 3 are independent, then:

$$C = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \text{Col} & \text{Col 2} & \text{Col 3} \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \quad \therefore R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{A}{\begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \text{Col 1} & \text{Col 2} & \text{Col 3} \\ \downarrow & \downarrow & \downarrow \end{bmatrix}} = \underset{C}{\begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \text{Col 1} & \text{Col 2} & \text{Col 3} \\ \downarrow & \downarrow & \downarrow \end{bmatrix}} \underset{R}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

Since R is 3×3 and A is 3×3, Matrix A has an RREF that is 3×3 (R_0) then R_0 does not have any zero rows and $R_0 = R$.

Given that the rank of Matrix A is 3 and Matrix A is 3×3, matrix A is full rank and $A\vec{x} = \vec{b}$ has 1 solution, and that solution corresponds to $\vec{x} = A^{-1}\vec{b}$

Note 5: A 2×2 Matrix given by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if $ad - bc \neq 0$, and its inverse is given by

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad (ad-bc \text{ is the determinant})$$

$AA^{-1} = I$ ✓

$A^{-1}A = I$ ✓

Note 6: The product of two matrices AB has an inverse if and only if A and B are separately invertible and the inverse of AB is $B^{-1}A^{-1}$.

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A(I)A^{-1} = AA^{-1} = I \checkmark$$
$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}(I)B = B^{-1}B = I \checkmark$$

By extension

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

Example

Elimination Matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R_2 - 5R_1$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_1$$

To obtain E_1^{-1}

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R_2 + 5R_1$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_1^{-1}$$

Check $E_1E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Check $E_1^{-1}E_1 = \cdots = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Can we use this in some way to find the inverse of Matrix A?

Finding the Inverse Using Gaussian Elimination

With GE, a total of 3 Elementary Operations are permitted in order to bring $A\vec{x} = \vec{b}$ to its RREF:

I: Interchange 2 Rows
II: Multiply One Row by a Nonzero Constant
III: Replace a Row by that Row Plus or Minus a Multiple of Another Row

I $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \dashrightarrow \begin{bmatrix} c & d \\ a & b \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

II $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow \begin{bmatrix} a & b \\ 8c & 8d \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 8 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 8c & 8d \end{bmatrix}$$

III $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow \begin{bmatrix} a+5c & b+5d \\ c & d \end{bmatrix}$

$$\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+5c & b+5d \\ c & d \end{bmatrix}$$

These are examples of Elementary Matrices.

Algorithm for finding A^{-1} (If A^{-1} Exists)

$$[A \mid I]$$

Start applying GE to bring Matrix A to its RREF

$$\begin{aligned} &\longrightarrow [E_1A \mid E_1I] \\ &\longrightarrow [E_2E_1A \mid E_2E_1I] \\ &\quad \vdots \\ &\longrightarrow [E_kE_{k-1}\cdots E_2E_1A \mid E_kE_{k-1}\cdots E_2E_1I] \end{aligned}$$

If after k Elementary Operations $E_kE_{k-1}\cdots E_2E_1A = I$, then we know that A^{-1} exists.

We can also see that $E_k E_{k-1} \cdots E_2 E_1 = A^{-1}$ and we now have an algorithm for finding A^{-1}.

$$[A \mid I] \xrightarrow{GE} [I \mid A^{-1}]$$

So not only have we used GE to find A^{-1}, we have also used it to express A^{-1} as a product of elementary matrices.

What about Matrix A?

$$A^{-1} = E_k E_{k-1} \cdots E_2 E_1$$

Every elementary matrix (E) is invertible and E^{-1} can be found by applying the reverse of the elementary operation that produced E to I. Furthermore, E^{-1} is also an elementary matrix.

$$(A^{-1})^{-1} = A = (E_k E_{k-1} \cdots E_2 E_1)^{-1}$$
$$= E_1^{-1} E_2^{-1} \cdots E_{k-1}^{-1} E_k^{-1}$$

$\therefore$ Every invertible Matrix A can be expressed as a product of elementary matrices.

An Example and a Segue into MAT185S

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Super Augmented Matrix

$$\left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] R_3 + 3R_4 \qquad E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\longrightarrow \left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] R_2 + 2R_3 \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\longrightarrow \left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 2 & 6 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] R_1 + R_2 \qquad E_3 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\longrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 2 & 6 \\ 0 & 1 & 0 & 0 & 0 & 1 & 2 & 6 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$= [I \mid A^{-1}]$$

$$AA^{-1} = I$$

$$E_3 E_2 E_1 A = I$$

$$\therefore A^{-1} = E_3 E_2 E_1$$

$$\therefore A = E_1^{-1} E_2^{-1} E_3^{-1}$$

From our earlier work we know R_0 (RREF of Matrix A)

$$R_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since R_0 has no zero rows, $R = R_0$

$$\underset{A}{\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}} = \underset{C}{\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \underset{R}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

$\therefore$ The 4 column vectors of matrix A form an independent set of vectors.

We will make a number of equivalent statements

a) Matrix A is invertible
b) $A\vec{x} = \vec{b}$ has one and only one solution (Unit 20)
c) $A\vec{x} = \vec{0}$ has only the trivial solution (Unit 20)
d) The RREF of Matrix A is I (Unit 20)
e) Matrix A is expressible as a product of elementary matrices (Unit 21)
f) The column vectors of Matrix A are independent (Unit 20, 22)