ESC103 - ENGINEERING MATHEMATICS and COMPUTATION

Numerical Solutions to First Order Differential Equations

Initial Value Problems (IVPs)

Example:

$\frac{dy(t)}{dt} = y'(t) = Cy(t)$ where c is a scalar

and

$y(t=0) = y(0)$

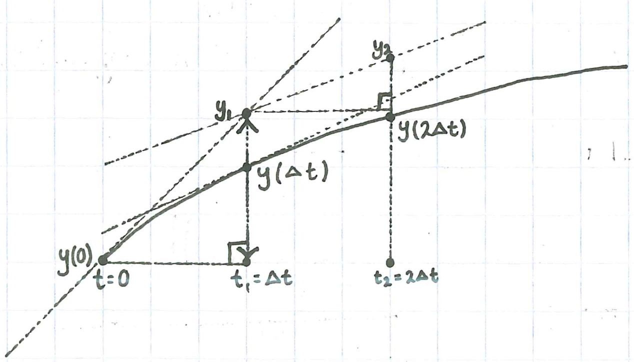
Try: $y(t) = y(0)e^{ct}$

$$y'(t) = C\underbrace{y(0)e^{ct}}_{y(t)} = Cy(t) \checkmark$$

$$y(t=0) = y(0)e^{c(0)} = y(0) \checkmark$$

Euler's Method

Given $y'(t) = f(t, y(t))$ and $y(t=0) = y(0)$ Solve for $y(t)$



$$y_1 = y(0) + \underbrace{\Delta t\, y'(t=0)}_{\text{Run}\cdot\text{Slope}} = y(0) + \Delta t f(0, y(0))$$

$$y_2 = y_1 + \Delta t\, y'(t=\Delta t) = y_1 + \Delta t f(\Delta t, y_1)$$

$$y_3 = y_2 + \Delta t\, y'(t=2\Delta t) = y_2 + \Delta t f(2\Delta t, y_2)$$

$\vdots$

Algorithm

$$t_{n+1} = t_n + \Delta t$$
$$y_{n+1} = y_n + \Delta t f(t_n, y_n)$$

$y_1, y_2, y_3 \ldots$ are estimates of the true solution

$y(t_1 = \Delta t), y(t_2 = 2\Delta t), y(t_3 = 3\Delta t) \ldots$

Sources of Error

S_1: The formula used to go from (t_n, y_n) to (t_{n+1}, y_{n+1}) is not exact.
S_2: The information going into the formula is not exact because (t_n, y_n) does not in general lie on the true solution curve except for $(t = 0, y(0))$.

Improved Euler's Method

$$t_{n+1} = t_n + \Delta t$$
$$y_{n+1} = y_n + \Delta t S$$

$$S = \frac{1}{2}(S_L + S_R)$$

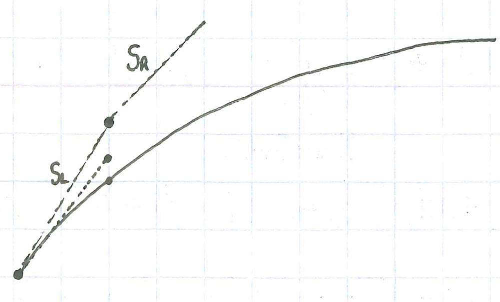
S_L = Slope Estimate at t_n
S_R = Slope Estimate at t_{n+1}

But we need y_{n+1} to determine S_R

$$t_{n+1} = t_n + \Delta t$$

$$y_{n+1,EM} = y_{n,IEM} + \Delta t f(t_n, y_{n,IEM})$$

$$y_{n+1,IEM} = y_{n,IEM} + \frac{\Delta t}{2}\Big(\underbrace{f(t_n, y_{n,IEM})}_{S_L} + \underbrace{f(t_{n+1}, y_{n+1,EM})}_{S_R}\Big)$$



Higher Order Systems

Consider the IVP

$$\frac{dx(t)}{dt} = x'(t) = a x(t) + b y(t) \qquad x(t=0) = x(0)$$

$$\frac{dy(t)}{dt} = y'(t) = c x(t) + d y(t) \qquad y(t=0) = y(0)$$

a, b, c, d are scalars

Let $z = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

$$\therefore z' = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}$$

$z' = Az$

$$\underset{(2\times 1)}{\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}} = \underset{(2\times 2)}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \underset{(2\times 1)}{\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}}$$

and

$$z_0 = \begin{bmatrix} x(0) \\ y(0) \end{bmatrix}$$

Euler's Method

$$t_{n+1} = t_n + \Delta t$$
$$z_{n+1} = z_n + \Delta t A z_n$$

Improved Euler's Method

$$t_{n+1} = t_n + \Delta t$$
$$z_{n+1,EM} = z_{n,IEM} + \Delta t A z_n$$
$$z_{n+1,IEM} = z_{n,IEM} + \frac{\Delta t}{2}\left(\underbrace{A z_n}_{S_L} + \underbrace{A z_{n+1,EM}}_{S_R}\right)$$

Consider

$$\frac{d^2 y(t)}{dt^2} = y''(t) = f(t, y(t), y'(t))$$

and

$y(t=0) = y(0)$

$y'(t=0) = y'(0)$

Solve $y(t)$

$$Z = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} \qquad Z' = \begin{bmatrix} y'(t) \\ y''(t) \end{bmatrix} \qquad Z_0 = \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}$$

Example $\quad y''(t) = -y(t) = f(t, y(t), y'(t))$

and

$y(0) = 1$

$y'(0) = 0$

$Z' = AZ$

$$\begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}$$

Euler's Method

$\longrightarrow Z_{n+1} = Z_n + \Delta t \, A Z_n$

$$\longrightarrow \begin{bmatrix} y_{n+1} \\ y'_{n+1} \end{bmatrix} = \begin{bmatrix} y_n \\ y'_n \end{bmatrix} + \Delta t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} y_n \\ y'_n \end{bmatrix}$$

$$\longrightarrow \begin{pmatrix} y_{n+1} = y_n + \Delta t \, y'_n \\ y'_{n+1} = y'_n + \Delta t(-y_n) \end{pmatrix}$$

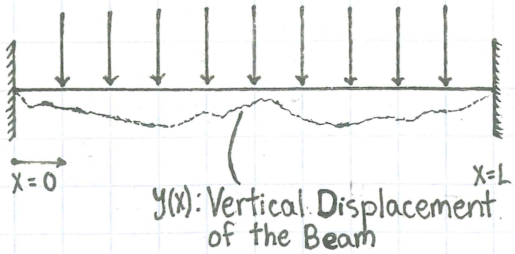
Boundary Value Problems (BVP)

$y'' + 2y' + y = 0$ on the interval $[0, 1]$

and

$y(0) = 0, \; y(1) = 1$

$$\frac{d^2 y}{dx^2} + \frac{2dy}{dx} + y(x) = 0$$



W(x): Linear Load Density

$$y''''(x) = \frac{W(x)}{EI}$$

E: Young's Modulus
I: 2^{nd} Moment of Beam's Cross Section

A numerical approach to solving BVP's begins by partitioning the interval $[a,b]$ into n evenly spaced subintervals.

$a=x_0 \quad x_1 \quad x_2 \quad \cdots \quad x_{n-1} \quad x_n=b$

Step Size $\Delta x = \frac{b-a}{n}$

The derivatives in the differential equation are approximated using finite differences.

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Forward Difference $\quad \Delta_F f(x) = \frac{f(x+\Delta x) - f(x)}{\Delta x}$

Backward Difference $\quad \Delta_B f(x) = \frac{f(x) - f(x-\Delta x)}{\Delta x}$

Central Difference $\quad \Delta_c f(x) = \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x}$

Example $\quad f''(x) \approx \Delta_B [\Delta_F f(x)]$

$$= \frac{\Delta_F f(x) - \Delta_F f(x-\Delta x)}{\Delta x}$$

$$= \frac{1}{\Delta x}\left[\frac{f(x+\Delta x) - f(x)}{\Delta x} - \frac{f(x) - f(x-\Delta x)}{\Delta x}\right]$$

$$= \frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{(\Delta x)^2}$$

Example (Unit 17)

$$\frac{dy(t)}{dt} = y'(t) = -4e^{-2t} + e^{t} = f(t, y(t))$$

and

$$y(t=0) = y(0) = 3$$

Euler's Method:

$$t_{n+1} = t_n + \Delta t$$
$$y_{n+1} = y_n + \Delta t\, f(t_n, y_n)$$
$$y_{n+1} = y_n + \Delta t(-4e^{-2t_n} + e^{t_n})$$

Example (BVP)

Given $y'' + 2y' + y = 0 \quad [0,1]$

and

$y(0) = 0,\ y(1) = 1$

$$y'(x) \approx \Delta_c\, y(x)$$
$$y''(x) \approx \Delta_B \Delta_F\, y(x)$$

$$\frac{y(x+\Delta x) - 2y(x) + y(x-\Delta x)}{(\Delta x)^2} + 2\frac{y(x+\Delta x) - y(x-\Delta x)}{2\Delta x} + y(x) = 0$$

Simplify:

$$\left(\frac{1}{(\Delta x)^2} + \frac{2}{2\Delta x}\right) y(x+\Delta x) + \left(\frac{-2}{(\Delta x)^2} + 1\right) y(x) + \left(\frac{1}{(\Delta x)^2} - \frac{2}{2\Delta x}\right) y(x-\Delta x) = 0$$

Choose to divide the interval $[0,1]$ into 5 subintervals $(n=5)$

$$\Delta x = \frac{1-0}{5} = 0.2$$

$$30y(x+\Delta x) - 49y(x) + 20y(x-\Delta x) = 0$$

$x = 0.2$

$$30y(0.4) - 49y(0.2) \cancel{+ 20y(0)} \xrightarrow{0 \text{ (B.C.)}} = 0$$

$x = 0.4$

$$30y(0.6) - 49y(0.4) + 20y(0.2) = 0$$

$x = 0.6$

$$30y(0.8) - 49y(0.6) + 20y(0.4) = 0$$

$x = 0.8$

$$30\underset{1 \text{ (B.C)}}{\cancel{y(1)}} - 49y(0.8) + 20y(0.6) = 0$$

$A\vec{x} = \vec{b}$

$$\begin{bmatrix} -49 & 30 & 0 & 0 \\ 20 & -49 & 30 & 0 \\ 0 & 20 & -49 & 30 \\ 0 & 0 & 20 & -49 \end{bmatrix} \begin{bmatrix} y(0.2) \\ y(0.4) \\ y(0.6) \\ y(0.8) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -30 \end{bmatrix}$$

Solving for $\vec{x}$

$$\vec{x} = \begin{bmatrix} y(0.2) \\ y(0.4) \\ y(0.6) \\ y(0.8) \end{bmatrix} = \begin{bmatrix} 0.4493 \\ 0.7338 \\ 0.8990 \\ 0.9792 \end{bmatrix}$$

Exact Solution

$$y(x) = xe^{1-x}$$

$$y(0.2) = 0.4451$$
$$y(0.4) = 0.7288$$
$$y(0.6) = 0.8951$$
$$y(0.8) = 0.9771$$