ESC103 - ENGINEERING MATHEMATICS and COMPUTATION

Finding Matrix R to Solve $A\vec{x} = \vec{0}$ (Special Class of $A\vec{x} = \vec{b}$)

$A\vec{x} = \vec{0}$ is referred to as a homogeneous system.
Matrix is $m \times n$ (square ($m=n$) or rectangular ($m \neq n$)).
This system always has at least one solution.

Example:

Solve $2x_1 + 3x_2 - 4x_3 = 0$

$A\vec{x} = \vec{0}$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad A = \begin{bmatrix} 2 & 3 & -4 \end{bmatrix}$$

$$\underset{(1\times3)}{\begin{bmatrix} 2 & 3 & -4 \end{bmatrix}} \underset{(3\times1)}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{(1\times1)}{\begin{bmatrix} 0 \end{bmatrix}}$$

Find $A = CR$

$C = [2] \quad R = ?$

$$\underset{(1\times3)}{A} = \underset{(1\times1)}{C}\underset{(1\times3)}{R}$$

$$\begin{bmatrix} 2 & 3 & -4 \end{bmatrix} = \begin{bmatrix} 2 \end{bmatrix}\begin{bmatrix} 1 & \frac{3}{2} & -\frac{4}{2} \end{bmatrix} \quad R = \tfrac{1}{2}A$$

$\therefore$ rank $A = 1$

If we have matrix R, we can solve $A\vec{x} = \vec{0}$ by solving $R\vec{x} = \vec{0}$

$$R\vec{x} = \vec{0}$$
$$CR\vec{x} = C\vec{0}$$
$$A\vec{x} = \vec{0}$$

$$\begin{bmatrix} 1 & 1.5 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$x_1 + 1.5x_2 - 2x_3 = 0$$

→ Declare x_1 to be a leading variable
→ Declare x_2 and x_3 to be free variables

$X_1 = -1.5X_2 + 2X_3$

$X_2 = X_2$

$X_3 = X_3$

$$\therefore \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = X_2 \begin{bmatrix} -1.5 \\ 1 \\ 0 \end{bmatrix} + X_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

2 Special Solutions to $A\vec{x} = \vec{0}$

$X_2 = 1$ and $X_3 = 0 \Rightarrow \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} -1.5 \\ 1 \\ 0 \end{bmatrix}$

$X_2 = 0$ and $X_3 = 1 \Rightarrow \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

Summary: $n = 3$ (3 unknowns)
$r = 1$ (Rank of Matrix A)
$n - r = 3 - 1 = 2$ (2 Special Solutions to $A\vec{x} = \vec{0}$)

Example 2:

Solve $X_1 + X_2 + 3X_4 = 0$
$X_3 - 2X_4 = 0$

$A\vec{x} = \vec{0}$

$$\underset{(2\times4)}{\begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}} \underset{(4\times1)}{\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}} = \underset{(2\times1)}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

Now Find $A = CR$

$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $R = ?$ $(2\times4) = (2\times2)(2\times4)$

$A = CR$

$\therefore R = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix} = A$

$\therefore \text{rank}\, A = 2$

$R\vec{x} = \vec{0}$

$$X_1 + X_2 + 3X_4 = 0$$
$$X_3 - 2X_4 = 0$$

$\rightarrow$ Declare X_1 and X_3 as leading

$$X_1 = -X_2 - 3X_4$$
$$X_2 = X_2$$
$$X_3 = 2X_4$$
$$X_4 = X_4$$

$$\therefore \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = X_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + X_4 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

Summary: $n = 4$
$r = 2$
$n - r = 4 - 2 = 2$

Example 3:

$$X_1 + 2X_2 + X_3 = 0$$
$$2X_1 + 4X_2 + 5X_3 = 0$$
$$3X_1 + 6X_2 + 9X_3 = 0$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 5 \\ 3 & 6 & 9 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 5 \\ 3 & 6 & 9 \end{bmatrix} R_2 - 2R_1$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 3 \\ 3 & 6 & 9 \end{bmatrix} R_3 - 3R_1$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 6 \end{bmatrix} R_2 \cdot \frac{1}{3}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 6 \end{bmatrix} R_3 - 6R_2$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} R_1 - R_2$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = R_0$$

This matrix is called the Reduced Row Echelon Form (RREF) of Matrix A. Also known as Reduced Normal Form (RNF).

Matrix R_0 becomes Matrix R ($A = CR$) by removing the zero row.

$$\therefore R = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R\vec{x} = \vec{0}$$

$$\begin{aligned} X_1 + 2X_2 &= 0 \\ X_3 &= 0 \end{aligned}$$

X_1, X_3 Leading
X_2 Free

$$\begin{aligned} X_1 &= -2X_2 \\ X_2 &= X_2 \\ X_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = X_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$$\underset{(3\times3)}{A} = \underset{(3\times2)}{C}\,\underset{(2\times3)}{R}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 5 \\ 3 & 6 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 5 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Leading 1's tell you where the independent columns in A are and also the rank of A.

Summary: $n = 3$
$r = 2$
$n - r = 3 - 2 = 1$

Solving $A\vec{x}=\vec{b}$ Using Gaussian Elimination (GE)

Let's begin by defining the properties of a matrix that has been brought to its Reduced Row Echelon Form called R_0.

P_I : The first non-zero entry in each row is a 1
P_2 : The other entries in the column containing these leading 1's (above and below) are zero
P_3 : The leading 1's move to the right as we move down the rows
P_4 : Any and all zero rows are collected at the bottom

Example:

Let's say A is a 4×5 matrix.

4 Equations
5 Unknowns

$$R = \begin{bmatrix} 1 & p & 0 & q & 0 \\ 0 & 0 & 1 & s & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (3 \times 5)$$

R_0 had one row of zeros
$\text{rank}(A) = 3$
1, 3, and 5 are the independent columns in Matrix A
X_1, X_3, X_5 are leading
X_2, X_4 are free

$$\underset{(4\times5)}{A} = \underset{(4\times3)}{C}\;\underset{(3\times5)}{R}$$

Within this algorithm, what operations are permitted for finding R_0 from Matrix A?

I : Interchange two rows (No Big Deal)
II : Multiply one row by a nonzero constant (NBD)
III : Add or subtract a multiple of one row to or from another row (Elimination)

These permitted operations are called elementary row operations and the algorithm used is called Gaussian Elimination (GE).

With $A\vec{x}=\vec{b}$ systems, we want to be able to find and write solutions like we did in Unit 14 but with $\vec{b}\neq\vec{0}$.

$$A\vec{x}=\vec{b} \xrightarrow{GE} R_0\vec{x}=\vec{d}$$

or in augmented matrix form:

$$[A\,|\,\vec{b}\,] \xrightarrow{GE} [R_0\,|\,\vec{d}\,]$$

Restricting ourselves to the permitted elementary row operations, $R_0\vec{x}=\vec{d}$ is called an equivalent system to $A\vec{x}=\vec{b}$ in that it has the same solution $\vec{x}$.

Example:

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 7 \end{bmatrix} \qquad A\vec{x}=\vec{b}$$

Augmented Matrix

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 1 & 3 & 1 & 6 & 7 \end{array}\right] R_3 - R_2$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 1 & 4 & 6 \end{array}\right] R_3 - R_2$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] = [R_0\,|\,\vec{d}\,]$$

The last equation is:

$$0x_1 + 0x_2 + 0x_3 + 0x_4 = 0$$
$$0 = 0$$

$\therefore$ All values for x_1, x_2, x_3, x_4 satisfy this equation.

What are the remaining equations and can they be solved in a similar fashion as found in Unit 14?

$\longrightarrow$ Declare x_1 and x_3 as leading variables

$\longrightarrow$ Declare x_2 and x_4 as free variables

$X_1 = 1 - 3X_2 - 2X_4$
$X_2 = X_2$
$X_3 = 6 - 4X_4$
$X_4 = X_4$

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 6 \\ 0 \end{bmatrix}}_{\text{A particular Solution } (X_2 = X_4 = 0)} + \underbrace{X_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + X_4 \begin{bmatrix} -2 \\ 0 \\ -4 \\ 1 \end{bmatrix}}_{\text{Solution to } A\vec{x} = \vec{0}}$$

Example:

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 6 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 1 & 3 & 1 & 6 & 6 \end{array}\right] R_3 - R_1$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 1 & 4 & 5 \end{array}\right] R_3 - R_2$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & -1 \end{array}\right] = [R_0 \mid \vec{d}]$$

The last equation is:

$$0X_1 + 0X_2 + 0X_3 + 0X_4 = -1$$

No values for X_1, X_2, X_3, X_4 satisfy this equation.

$\therefore$ This system has no solution.

We did not see this in Unit 14 when solving $A\vec{x} = \vec{0}$ because in this case $\vec{x} = \vec{0}$ is always a solution.

Connecting Rank and the Shape of a Matrix

When using GE to help solve $A\vec{x} = \vec{b}$, there are 3 possible outcomes:

1. Unique Solution
   - All the variables are leading variables (no free variables)
2. Infinitely Many Solutions
   - At least one free variable
3. No Solution

$$[0\ 0\ 0 \cdots 0 \mid *]$$

Case 1 and 2 are consistent systems.
Case 3 is an inconsistent system.

The RREF of Matrix A is unique.
The number of leading 1's in the RREF of Matrix A corresponds to the rank of Matrix A.

Shapes of Matrices

Square Systems ($m = n = 2$)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \xrightarrow{GE} [R_0 \mid \vec{d}\,]$$

$$\Updownarrow$$

$$[R_0]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

Possible R_0

$$R_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad R_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$R_0 = \begin{bmatrix} 1 & * \\ 0 & 0 \end{bmatrix} \quad R_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Rank can be 1 or 2. When $r = 1 \Rightarrow 0$ or ∞ solutions (Not Full Rank)
$r = 2 \Rightarrow 1$ solution (Full Rank)

Tall and Thin Systems ($m > n$, $m = 3$ $n = 2$)

$$\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Possible R_0

$$R_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad R_0 = \begin{bmatrix} 1 & * \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad R_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Rank can be 1 or 2. When $r = 1 \Rightarrow 0$ or ∞ solutions (Not Full Rank)
$r = 2 \Rightarrow 0$ or 1 solution (Full Rank)

Short and Wide Systems ($m<n$, $m=2$ $n=3$)

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Possible R_0

$$R_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad R_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad R_0 = \begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \end{bmatrix} \quad R_0 = \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \end{bmatrix}$$

Rank can be 1 or 2. When $r=1 \Rightarrow 0$ or ∞ solutions (Not Full Rank)
$r=2 \Rightarrow \infty$ solutions (Full Rank)

Summary

Full Rank

Square ($m=n$) $r=m, r=n$ $A\vec{x}=\vec{b}$ 1 solution
Tall and Thin ($m>n$) $r<m, r=n$ $A\vec{x}=\vec{b}$ 0 or 1 solution
Short and Wide ($m<n$) $r=m, r<n$ $A\vec{x}=\vec{b}$ ∞ solution

Not Full Rank

Any System $r<m, r<n$ $A\vec{x}=\vec{b}$ 0 or ∞ solutions