ESC103 - ENGINEERING MATHEMATICS and COMPUTATION

Solving Linear Equations Using Elimination

$A\vec{x} = \vec{b}$

For now, we will assume we have n equations with n unknowns.

$A\vec{x} = \vec{b} \xrightarrow{\text{elimination}} U\vec{x} = \vec{c}$

U: Denotes an upper triangular matrix where only zeros appear below the diagonal.

Example:

$$2x_1 + 3x_2 + 4x_3 = b_1$$
$$4x_1 + 11x_2 + 14x_3 = b_2$$
$$2x_1 + 8x_2 + 17x_3 = b_3$$

$A\vec{x} = \vec{b}$

$$\begin{bmatrix} 2 & 3 & 4 \\ 4 & 11 & 14 \\ 2 & 8 & 17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Lets focus our attention initially on just Matrix A.

Pivot

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 11 & 14 \\ 2 & 8 & 17 \end{bmatrix} R_2 - 2R_1$$

$$\rightarrow \begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 2 & 8 & 17 \end{bmatrix} R_3 - R_1$$

Pivot

$$\rightarrow \begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 5 & 13 \end{bmatrix} R_3 - R_2$$

$$\rightarrow \begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{bmatrix}$$

$$\therefore U = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{bmatrix} \Bigg\}$$ What we've really done is rotate two of the planes

$A\vec{x}=\vec{b} \quad [A|\vec{b}]$

$$\left[\begin{array}{ccc|c} 2 & 3 & 4 & b_1 \\ 4 & 11 & 14 & b_2 \\ 2 & 8 & 17 & b_3 \end{array}\right] R_2-2R_1$$

$$\longrightarrow \left[\begin{array}{ccc|c} 2 & 3 & 4 & b_1 \\ 0 & 5 & 6 & b_2-2b_1 \\ 2 & 8 & 17 & b_3 \end{array}\right] R_3-R_1$$

$$\longrightarrow \left[\begin{array}{ccc|c} 2 & 3 & 4 & b_1 \\ 0 & 5 & 6 & b_2-2b_1 \\ 0 & 5 & 13 & b_3-b_1 \end{array}\right] R_3-R_2$$

$$\longrightarrow \left[\begin{array}{ccc|c} 2 & 3 & 4 & b_1 \\ 0 & 5 & 6 & b_2-2b_1 \\ 0 & 0 & 7 & b_3-b_1-b_2+2b_1 \end{array}\right]$$

$$= \left[\begin{array}{c|c} U & \begin{array}{c} C_1 \\ C_2 \\ C_3 \end{array} \end{array}\right]$$

$C_1=b_1, \; C_2=b_2-2b_1, \; C_3=b_1-b_2+b_3$

$U\vec{x}=\vec{c}$

Assume $\vec{c}=\begin{bmatrix} 19 \\ 17 \\ 14 \end{bmatrix}$

$$\begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 19 \\ 17 \\ 14 \end{bmatrix}$$

$$7x_3=14 \Rightarrow x_3=2$$
$$5x_2+6(2)=17 \Rightarrow x_2=1$$
$$2x_1+3(1)+4(2)=19 \Rightarrow x_1=4$$

It turns out that every matrix A with independent columns can be reduced to an upper triangular matrix U with non zero pivots.

Elimination Matrices

We can achieve what we did in Unit II (Solving Linear Equations Using Elimination) by multiplying matrix A by another matrix.

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -3 & 1 & 1 \\ 6 & 8 & 4 \end{bmatrix} R_2 + R_1$$

$$\longrightarrow \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 1 \\ 6 & 8 & 4 \end{bmatrix}$$

Elimination Matrix E_1

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 0 \\ -3 & 1 & 1 \\ 6 & 8 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 1 \\ 6 & 8 & 4 \end{bmatrix}$$

How to find E_1?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R_2 + R_1$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_1$$

Recall:

$$\begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 1 \\ 6 & 8 & 4 \end{bmatrix} R_3 - 2R_1$$

$$\longrightarrow \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 6 & 4 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$E_2(E_1A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 1 \\ 6 & 8 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 6 & 4 \end{bmatrix} R_3 - 3R_2$$

$$\longrightarrow \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

$$E_3(E_2E_1A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 6 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} = U$$

$\therefore E_3E_2E_1A = U$

or $EA = U$ where $E = E_3E_2E_1$

What does E Equal?

$$E = (E_3E_2)E_1$$

$$E_3E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -3 & 1 \end{bmatrix}$$

$$(E_3E_2)E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -5 & -3 & 1 \end{bmatrix}$$

$$\therefore E = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -5 & -3 & 1 \end{bmatrix}$$

$$\underset{\uparrow \atop E}{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -5 & -3 & 1 \end{bmatrix}}\underset{\uparrow \atop A}{\begin{bmatrix} 3 & 1 & 0 \\ -3 & 1 & 1 \\ 6 & 8 & 4 \end{bmatrix}} = \underset{\uparrow \atop U}{\begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}} = U$$