ESC103 - ENGINEERING MATHEMATICS and COMPUTATION

Matrix Multiplication

We denote the multiplication of 2 matrices A and B by:

$$AB$$

We can do matrix multiplication by an extension of how we defined $A\vec{x}$ but in doing so, this places certain conditions on A and B, when A has n columns, matrix B must have n rows.

A is $m \times n$ and B is $n \times p$

$$(m \times n)(n \times p)$$

The internal dimensions (n and n) must match and the external dimensions (m and p) dictate the dimensions of the product ($m \times p$).

Method 1: Combinations of Columns of Matrix A

Let the columns of B be given by

$$B = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \vec{x} & \vec{y} & \vec{z} \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$$

Then the columns of AB are given by

$$AB = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ A\vec{x} & A\vec{y} & A\vec{z} \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$$

Example:

$$AB = \underset{(2\times 2)}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} \underset{(2\times 2)}{\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}}$$

$$= \begin{bmatrix} A\begin{bmatrix} 5 \\ 7 \end{bmatrix} & A\begin{bmatrix} 6 \\ 8 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 5\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 7\begin{bmatrix} 2 \\ 4 \end{bmatrix} & 6\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 8\begin{bmatrix} 2 \\ 4 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 5 \\ 15 \end{bmatrix} + \begin{bmatrix} 14 \\ 28 \end{bmatrix} & \begin{bmatrix} 6 \\ 18 \end{bmatrix} + \begin{bmatrix} 16 \\ 32 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

Method 2: Use of Dot Products

(Row i of Matrix A) $\cdot$ (Column j of Matrix B)
Goes into Row i, Column j Entry of AB

Example:

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

$$AB = \begin{bmatrix} (\text{Row 1 of A}) \cdot (\text{Column 1 of B}) & (\text{Row 1 of A}) \cdot (\text{Column 2 of B}) \\ (\text{Row 2 of A}) \cdot (\text{Column 1 of B}) & (\text{Row 2 of A}) \cdot (\text{Column 2 of B}) \end{bmatrix}$$

$$AB = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

Both methods require 8 multiplications.

The Identity Matrix and The Exchange Matrix

$$I_{2\times2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ (Identity Matrix)}$$

$$AI = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A$$

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A$$

$$Ex = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ (Exchange Matrix)}$$

$$AEx = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

$$ExA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

$$AEx \neq ExA$$

In general, $AB \neq BA$ (Not Commutative).

Matrix multiplication is Associative.

$$ABC$$
$$(AB)C = A(BC)$$

Multiplying matrices with incompatible dimensions will be undefined.

$$(7 \times 2)(2 \times 4) \longrightarrow (7 \times 4)$$
$$(2 \times 4)(7 \times 2) \longrightarrow \text{Undefined}$$

Supplement to Unit 9 (Matrix Multiplication)

Let $A=[a_{ij}] \in R^{m\times n}$
$B=[b_{jk}] \in R^{n\times p}$

then the product of matrix multiplication of A and B, denoted by

$$AB = C = [C_{ik}] \in R^{m\times p} \qquad (m\times n)(n\times p)$$

C is defined as

$$C_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$$
$$= a_{i1}b_{1k} + a_{i2}b_{2k} + \cdots + a_{in}b_{nk}$$

The i,k entry in $C=AB$ that is found in the i^{th} row and k^{th} column is

(row i of A) · (Column k of B)

Factoring Matrix A=CR

Step 1: Construct Matrix C from Matrix A

→ If column 1 of matrix A is <u>Not</u> all zeros, put it into matrix C
→ If column 2 of matrix A is <u>Not</u> some multiple of column 1, put it into matrix C
→ If column 3 of matrix A is <u>Not</u> some linear combination of column 1 and column 2, put it into Matrix C
→ Continue for all columns of matrix A

At the end of step 1, matrix C will have r columns taken from matrix A ($r \leq n$)

The original columns of matrix A might be dependent, but the r columns of matrix C will be independent.

Given that the independent columns of matrix C combine to give all columns of matrix A, we have what we call in linear algebra a basis for the column space of matrix A.

This number r is called the <u>rank</u> of matrix A.

Example:

$$A = \begin{bmatrix} 2 & 6 & 4 \\ 4 & 12 & 8 \\ 1 & 3 & 5 \end{bmatrix}$$

$$\therefore C = \begin{bmatrix} 2 & 4 \\ 4 & 8 \\ 1 & 5 \end{bmatrix} \qquad r(A) = 2$$

$$A = CR$$

$$\begin{bmatrix} 2 & 6 & 4 \\ 4 & 12 & 8 \\ 1 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 4 & 8 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} \times & \triangle & \square \\ \times & \triangle & \square \end{bmatrix}$$

$$(3 \times 3) \qquad (3 \times 2) \qquad (2 \times 3)$$

$$R = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 6 \\ 13 \\ 3 \end{bmatrix} = (3) \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} + (0) \begin{bmatrix} 4 \\ 8 \\ 5 \end{bmatrix}$$

<u>Summary</u>

⟶ Matrix C contains r independent columns of matrix A
⟶ r is the rank of matrix A r(A)
⟶ Matrix R tells us how to combine the columns of matrix C to give <u>All</u> columns in matrix A

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = c \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + d \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = (-1) \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} \qquad r(A) = 2$$

$$(3 \times 3) \qquad (3 \times 2) \qquad (2 \times 3)$$